PRODUCTS OF FITTING CLASSES OF FINITE GROUPS

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ABSTRACT

Using Fitting classes we generalize some well known theorems on centralizers in finite groups.

1. Introduction

In this work, a group means a finite group. A nonempty class of finite groups θ is called a Fitting class if: it is homomorphism-invariant, for every $G \in \theta$ every normal subgroup of G is a θ-group, and the product of normal θ-subgroups of an arbitrary group G belongs to θ. A group G is called a θ-group if $G \in \theta$. In particular, the product $F_\theta(G)$ of all normal θ-subgroups of G is a θ-group.

Let θ and θ' be Fitting classes and denote by $\theta\theta'$ the class of groups G such that $G/F_\theta(G) \in \theta'$. We prove in Theorem 2.9 that $\theta\theta'$ is Fitting class. Now define:

$$\theta_n = \begin{cases} \theta & \text{for } n \text{ odd} \\ \theta' & \text{for } n \text{ even} \end{cases} \qquad \text{and} \qquad \begin{aligned} (\theta\theta')_2 &= \theta\theta' \\ (\theta\theta')_n &= (\theta\theta')_{n-1}\theta_n \end{aligned}$$

We obtain that $F_{(\theta\theta')_n}(G)/F_{(\theta\theta)_{n-1}}(G) = F_{\theta_n}(G/F_{(\theta\theta')_{n-1}}(G))$ (Corollary 2.14).

If $F_\theta(G/F_\theta(G)) = 1$, we shall say that G has θ-length of order 1. A finite group is called $\theta\theta'$-separable if every composition factor of G is either a θ-group or a θ'-group. Let γ_π denote the Fitting class of π-groups. Now if we set $\theta_\pi = \theta \cap \gamma_\pi$ we may express our principal result as follows:

THEOREM A. *If $C_G(F_{\theta_\pi}(G))$ is $\theta_\pi\gamma_{\pi'}$-separable with θ_π-length of order* 1, *then $O_{\pi'}(G) = 1$ implies that $C_G(F_{\theta_\pi}(G)) \subseteq F_{\theta_\pi}(G)$.*

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Various corollaries of Theorem A generalize well known theorems of group theory.

In Corollary 4.7, we prove that if $C_G(O_{(\pi'\pi)_n}(G))$, $n \geqq 2$, is π-separable then $C_G(O_{(\pi'\pi)_n}(G)) \subseteq O_{(\pi'\pi)_n}(G)$. For $n = 2$ we obtain that if $C_G(O_{\pi'\pi}(G))$ is π-separable then $C_G(O_{\pi'\pi}(G)) \subseteq O_{\pi'\pi}(G)$. In particular we obtain Lemma 1.2.3 of Hall-Higman [3]: If $C_G(O_\pi(G))$ is π-separable and $O_{\pi'}(G) = 1$ then $C_G(O_\pi(G)) \subseteq O_\pi(G)$.

Let ν denote the Fitting class of nilpotent groups. Corollary 4.8 says that if $C_G(F_{(\gamma_{\pi'}\nu_\pi)_n}(G))$, $n \geqq 2$, is π-solvable, then $C_G(F_{(\gamma_{\pi'}\nu_\pi)_n}(G)) \subseteq F_{(\gamma_{\pi'}\nu_\pi)_n}(G)$. In particular, since $O_{\pi'}(G) = 1$ implies that $F_{\nu_\pi}(G) = F(G)$, we obtain (when $F(G)$ is a Fitting subgroup of G): if $C_G(F(G))$ is π-solvable and $O_{\pi'}(G) = 1$ then $C_G(F(G)) \subseteq F(G)$. In addition, if we define $F^1(G) = F(G)$ and $F^n(G)/F^{n-1}(G) = F(G/F^{n-1}(G))$, then Corollary 4.8 yields the following result: if $C_G(F^n(G))$ is solvable then $C_G(F^n(G)) \subseteq F^n(G)$.

The last two corollaries generalize the well-known theorem that if G is solvable then $C_G(F(G)) \subseteq F(G)$.

Finally let K be an S_π-subgroup of the π-solvable group G, i.e., K is a π-group and $|G:K|$ is divisible by no primes in π. We prove in Theorem 4.12 that $C_G(K \cap F_{\gamma_{\pi'}\nu_\pi}(G)) \subseteq F_{\gamma_{\pi'}\nu_\pi}(G)$. This result generalizes, to π-solvable groups, the theorem that if P is an S_p-subgroup of the p-solvable group G, then
$C_G(P \cap O_{p'p}(G)) \subseteq O_{p'p}(G)$.

All these corollaries are derived from Theorem A by considering various Fitting classes.

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2. On Fitting classes

It is assumed that the reader is familiar with the basic notation of [1] and [2].

DEFINITION 2.1. Let θ be a nonempty class of finite groups. If $G \in \theta$, we shall say that G is a *θ-group*.

The class θ is a *Fitting Class* if the following conditions hold:

(i) Epimorphic images of θ-groups are θ-groups.

(ii) Normal subgroups of θ-groups are θ-groups.

(iii) The product of normal θ-subgroups of a group G is a θ-group.

Condition (i) does not appear in the definition of [1] and [4]. Since we assume that θ-groups exist, it follows by (i) that $1 \in \theta$.

DEFINITION 2.2. Let γ_π denote the class of π-groups, and ν the class of nilpotent groups.

EXAMPLE 2.3. Both γ_π and ν are Fitting classes.

One can now form the characteristic subgroup $F_\theta(G)$ of G, *which is the product of all normal θ-subgroups of G*. Note that if θ is the class ν then $F_\nu(G) = F(G)$ is just the Fitting subgroup of G, and if θ is the class γ_π then $F_{\gamma_\pi}(G) = O_\pi(G)$.

DEFINITION 2.4. Let θ, θ' be Fitting classes. Then $\theta\theta'$ is the class of groups G such that $G/F_\theta(G) \in \theta'$.

We wish to prove that $\theta\theta'$ is a Fitting class. First, we state four Lemmas, which follow easily from the definitions.

LEMMA 2.5. *If θ, θ' are Fitting classes then $\in \theta\theta'$ if and only if there is a normal subgroup $N \in \theta$ of G such that $G/N \in \theta'$.*

LEMMA 2.6. *If θ is a Fitting class and N is a normal subgroup of a group G, then $F_\theta(G)N/N \subseteq F_\theta(G/N)$.*

LEMMA 2.7. *If V, W, M are subgroups of a group G such that $M \lhd V$ and $W \lhd G$, then VW/MW is an epimorphic image of V/M.*

LEMMA 2.8. *Let θ be a Fitting class and K a normal subgroup of a group G. Then $F_\theta(K) = K \cap F_\theta(G)$.*

The previous lemmas imply the following:

THEOREM 2.9. *Let θ, θ' be Fitting classes. Then $\theta\theta'$ is a Fitting class.*

Let $\theta, \theta', \theta'', \cdots, \theta^{(n)}, \cdots$ be Fitting classes. By Theorem 2.9 the symbols $F_{\theta\theta'}(G), F_{(\theta\theta')\theta''}$, etc., are well defined.

We shall write $\theta\theta'\theta'' \cdots \theta^{(n)}$ for $(\cdots((\theta\theta')\theta'')\cdots)\theta^{(n)}$.

LEMMA 2.10. *Let θ, θ' be Fitting classes. Then*

(i) $F_\theta(G) \subseteq F_{\theta\theta'}(G)$

(ii) $F_\theta(G) \subseteq F_{\theta'\theta}(G)$.

This follows at once from the definitions.

DEFINITION 2.11. Let θ, θ' be Fitting classes and let

$$\theta_n = \begin{cases} \theta & \text{for } n \text{ odd} \\ \theta' & \text{for } n \text{ even} \end{cases}$$

We define the Fitting Classes: $(\theta\theta')_1 = \theta$ $(\theta\theta')_2 = \theta\theta'$ and $(\theta\theta')_n = ((\theta\theta')_{n-1})\theta_n$, $n \geqq 2$.

Now, if $\theta_1, \theta_2, \theta_3$ are Fitting classes it is clear that $(\theta_1\theta_2)\theta_3 = \theta_1(\theta_2\theta_3)$. Therefore we shall write the Fitting class $(\theta\theta')_n$ by $\theta\theta' \cdots \theta_n$.

DEFINITION 2.12. Let θ, θ' be Fitting classes. By repeated applications of Lemma 2.10, we obtain a sequence of characteristic subgroups of G:

$$1 \subseteq F_\theta(G) \subseteq F_{\theta\theta'}(G) \subseteq F_{\theta\theta'\theta}(G) \subseteq \cdots.$$

This sequence is called the *$\theta\theta'$-Series* of G.

Using the above lemmas we get

THEOREM 2.13. *Let θ, θ' be Fitting classes. Then*

$$F_{\theta\theta'}(G)/F_\theta(G) = F_{\theta'}(G/F_\theta(G))$$

By induction we get (see def. 2.11.):

COROLLARY 2.14.

$$F_{(\theta\theta')_n}(G)/F_{(\theta\theta')_{n-1}}G) = F_{\theta_n}(G/F_{(\theta\theta')_{n-1}}(G)).$$

EXAMPLE 2.15. The $\gamma_\pi\gamma_{\pi'}$-series of G is the upper π-series of G. (For the definition of the upper π-series of G see [2], chap. VI).

3. The main theorem

We shall now study two basic classes of groups which generalize the classes of π-solvable groups and π-separable groups.

DEFINITION 3.1. Let θ, θ' be Fitting classes. We shall say that G is *$\theta\theta'$-separable* if every composition factor of G is either a θ-group or a θ'-group.

EXAMPLE 3.2. G is π-separable if and only if G is $\gamma_\pi\gamma_{\pi'}$-separable.

DEFINITION 3.3. Let θ, θ' be Fitting classes. We shall say that G is *$\theta\theta'$-solvable* if every composition factor is either a θ'-group or both an θ-group and a p-group for some prime p.

EXAMPLE 3.4. G is π-solvable if and only if G is $\gamma_\pi\gamma_{\pi'}$-solvable.

DEFINITION 3.5. Let $\mathrm{Sep}(\theta, \theta')$ denote the class of $\theta\theta'$-separable groups, and $\mathrm{Sol}(\theta, \theta')$ the class of $\theta\theta'$-solvable groups.

The definitions immediately yield:

LEMMA 3.6. *Let θ, θ' be Fitting classes, and let H be a normal subgroup of G such that K and G/K are both either $\mathrm{Sep}(\theta, \theta')$-groups or $\mathrm{Sol}(\theta, \theta')$-groups. Then G is a $\mathrm{Sep}(\theta, \theta')$-group or a $\mathrm{Sol}(\theta, \theta')$-group, accordingly.*

LEMMA 3.7. *Let θ, θ' be Fitting classes. Then* $\mathrm{Sep}(\theta, \theta')$ *and* $\mathrm{Sol}(\theta, \theta')$ *are Fitting classes.*

PROOF. Observe that as an immediate consequence of the definitions, homomorphic images and normal subgroups of $\theta\theta'$-separable groups are also $\theta\theta'$-separable.

Now, if H and K are normal $\theta\theta'$-subgroups of the group G, then Lemma 3.6 implies that HK is a normal $\theta\theta'$-separable subgroup of G. We obtain the conclusion for $\mathrm{Sol}(\theta, \theta')$ similarly.

LEMMA 3.8. *Let θ, θ' be Fitting classes, and let G be a $\theta\theta'$-separable group. Then:*

(i) *A minimal normal subgroup of G is either a θ-group or a θ'-group.*

(ii) $F_\theta(G) = F_{\theta'}(G) = 1$ *implies* $G = 1$.

(iii) *The $\theta\theta'$-series of G terminates with G.*

PROOF. (i) Let K be a minimal normal subgroup of G. Then K is characteristically simple, whence K is the direct product of isomorphic simple groups K_i, $1 \leqq i \leqq n$. Since $K \lhd G$, there exists a composition series of G in which K_1 is the last nontrivial term, whence K_1 is a composition factor of G. Since G is $\theta\theta'$-separable, K_1, and hence each K_i, is either a θ-group or a θ'-group, proving (i).

Statement (ii) follows immediately from (i).

(ii) Assume that the $\theta\theta'$-series of G terminates in a proper subgroup H of G. Thus, $H = F_{(\theta\theta')_n}(G) = F_{(\theta\theta')_{n+i}}(G) \subset G$ for all $i \geqq 1$. Corollary 2.14 implies therefore that $F_\theta(G/H) = F_{\theta'}(G/H) = \bar{1}$. Since by Lemma 3.7 G/H is $\theta\theta'$-separable, part (ii) yields $G/H = \bar{1}$, a contradiction.

DEFINITION 3.9. Let θ_π denote the Fitting class of the intersection of a Fitting class θ with γ_π, i.e., $\theta_\pi = \theta \cap \gamma_\pi$. (Note that $1 \in \theta_\pi$, and an intersection of Fitting classes is obviously a Fitting class.)

DEFINITION 3.10. Let θ be a Fitting class. We shall say that G has *θ-length* of order n if the $\theta\theta$-series of G terminates in $F_{(\theta\theta)_n}(G)$, i.e., $F_{(\theta\theta)_{n-1}}(G) \subset F_{(\theta\theta)_n}(G) = F_{(\theta\theta)_{n+i}}(G)$ for all $i \geqq 1$.

We state now our main result.

THEOREM A. *Let θ be a Fitting class and let the subgroup $C_G(F_{\theta_\pi}(G))$ of G be $\theta_\pi\gamma_{\pi'}$-separable, with θ_π-length of order* 1. *Then $O_{\pi'}(G) = 1$ implies that* $C_G(F_{\theta_\pi}(G)) \subseteq F_{\theta_\pi}(G)$.

PROOF. First we prove the following Lemma: If θ is a Fitting class, then

$F_\theta(C_G(F_\theta(G))) = Z(F_\theta(G))$. We have $C_G(F_\theta(G)) \cap F_\theta(G) = Z(F_\theta(G))$. Now $F_\theta(C_G(F_\theta(G)))$ char $C_G(F_\theta(G))$ char G, so $F_\theta(C_G(F_\theta(G))) \subseteq F_\theta(G)$. Consequently $F_\theta(C_G(F_\theta(G))) \subseteq F_\theta(G) \cap C_G(F_\theta(G)) = Z(F_\theta(G))$.

On the other hand $Z(F_\theta(G)) \lhd F_\theta(G)$ and $Z(F_\theta(G)) \subseteq C_G(F_\theta(G))$. Hence $Z(F_\theta(G)) \subseteq F_\theta(C_G(F_\theta(G)))$.

We proceed with the proof of the theorem. Set $F_{\theta_\pi}(G) = H$ and $C = C_G(H)$. We have to show that $C \subseteq H$ or equivalently $C = Z(H)$. We know from the above proved lemma that $F_{\theta_\pi}(C) = Z(H)$. Thus it suffices to prove that $C = F_{\theta_\pi}(C)$ Assume now that $C \supset F_{\theta_\pi}(C)$. Since C is $\theta_\pi\gamma_{\pi'}$-separable, the $\theta_\pi\gamma_{\pi'}$-series of C terminates with C by Lemma 3.8 (iii). By hypothesis $F_{\theta_\pi}(C/F_{\theta_\pi}(C)) = 1$. Since $C/F_{\theta_\pi}(C)$ is $\theta_\pi\gamma_{\pi'}$-separable, it follows by Lemma 3.8(ii) that $L = F_{\theta_\pi\gamma_{\pi'}}(C) \supset F_{\theta_\pi}(C)$ Since $L/F_{\theta_\pi}(C) = F_{\theta_\pi\gamma_{\pi'}}(C)/F_{\theta_\pi}(C) = O_{\pi'}(C/F_{\theta_\pi}(C))$ is a π'-group, $Z(H) = F_{\theta_\pi}(C)$ is a normal S_π-subgroup of L. Hence by the Schur-Zassenhaus theorem, $Z(H)$ possesses a complement $K \neq 1$ in L which is an $S_{\pi'}$-subgroup of L.

But $K \subseteq C$ and C centralizes $Z(H)$, whence $L = Z(H) \times K$ and as K is a π'-group, we have K char L char C char G. Thus $K \lhd G$ and consequently $K \subseteq O_{\pi'}(G) = 1$, a contradiction. It follows that $C = F_{\theta_\pi}(C)$, thus proving Theorem A.

4. Some consequences of the main theorem

Theorem A and the following lemmas and theorems yield important corollaries which generalize some well known theorems on finite groups.

DEFINITION 4.1. Let $\theta_1, \theta_2, \cdots, \theta_n$ be Fitting classes. We shall say that G is $\theta_1 \cdots \theta_n$-*separable* if every composition factor of G is θ_i-group for some i, $1 \leqq i \leqq n$.

NOTE 1. G is a $\theta_1 \cdots \theta_n$-group if $G \in \theta_1 \cdots \theta_n$, when $\theta_1 \cdots \theta_n$ is a Fitting class.

NOTE 2. If a composition factor of G is a $\theta_1 \cdots \theta_n$-group then, being simple, it is a θ_i-group for some i, $1 \leqq i \leqq n$.

Thus if θ is the Fitting class $\theta_1 \cdots \theta_n$, then G is θ-separable (i.e., every composition factor of G is θ-group) if and only if it is $\theta_1 \cdots \theta_n$-separable.

LEMMA 4.2. *Let θ be a Fitting class with the Z property,* i.e. $G/H \in \theta, H \subseteq Z(G)$ *implies that $G \in \theta$. Then $F_\theta(G/H) = F_\theta(G)/H$ for every $H \subseteq Z(G)$.*

PROOF. It is clear that $HF_\theta(G)/H \subseteq F_\theta(G/H) = R/H$. Now $R/H \in \theta$ implies $R \in \theta$. Therefore $R/H \subseteq F_\theta(G)/H$.

COROLLARY 4.3. *Let θ be a Fitting class with the Z property. Then $C_G(F_\theta(G))$ has θ-length of order 1.*

PROOF. Lemma 4.2 and the first part of the proof of Theorem A imply that: $F_\theta[C_G(F_\theta(G)/F_\theta(C_G(F_\theta(G)))] = F_\theta[C_G(F_\theta(G))/Z(F_\theta(G)] = F_\theta(C_G(F_\theta(G)))/Z(F_\theta(G)) = 1$.

LEMMA 4.4. *If θ and θ' are Fitting classes, and if θ satisfies the Z property, then $\theta\theta'$ satisfies the Z property.*

PROOF. Let $G/H \in \theta\theta'$ when $H \subseteq Z(G)$. By definition $[G/H]/[F_\theta(G/H)] \in \theta'$. Thus $[(G/H)]/[F_\theta(G)/H] \in \theta'$ by Lemma 4.2. Therefore $G/F_\theta(G) \in \theta'$ and we obtain $G \in \theta\theta'$.

LEMMA 4.5.

(i) $\gamma_\pi\gamma_{\pi'}$ *has the Z property.*

(ii) $\gamma_\pi\nu_{\pi'}$ *has the Z property.*

(iii) $\nu_\pi\nu_{\pi'}$ *has the Z property.*

PROOF. (i) Let $G/H \in \gamma_\pi\gamma_{\pi'}$ and $H \subseteq Z(G)$. By definition there exists a normal S_π-subgroup K/H of G/H. Since H is a nilpotent group, $H = H_\pi \times H_{\pi'}$ when H_π and $H_{\pi'}$ are the S_π and $S_{\pi'}$-subgroup of H, respectively. Thus $H_{\pi'}$ is a normal $S_{\pi'}$-subgroup of K, and an S_π-subgroup K_π of K exists. Since $H_{\pi'}$ centralizes K_π, $K = K_\pi \times H_{\pi'}$. Hence K_π is a normal S_π-subgroup of G, yielding $G \in \gamma_\pi\gamma_{\pi'}$.

(ii) Let $G/H \in \gamma_\pi\nu_{\pi'}$ and $H \subseteq Z(G)$. Part (i) implies that G has a normal S_π-subgroup K_π. Let R/H be the nilpotent $S_{\pi'}$-subgroup of G/H. H_π of part (i) is a normal S_π-subgroup of R. Hence $R = R_{\pi'} \times H_\pi$, where $R_{\pi'}$ is an $S_{\pi'}$-subgroup of R, hence of G. Since R/H is nilpotent and $H \subseteq Z(R)$ we obtain that R is nilpotent and so $R_{\pi'}$ is nilpotent. Thus $G \in \gamma_\pi\nu_{\pi'}$.

Now it is a simple matter to verify (iii).

THEOREM 4.6. *Let θ_i $1 \leqq i \leqq n$ be Fitting classes. Let $C_G(F_{\gamma_\pi\gamma_{\pi'}\theta_1\cdots\theta_n}(G))$ be $\gamma_\pi\gamma_{\pi'}\theta_1,\cdots\theta_n$-separable subgroup of a group G. Then*

$$C_G(F_{\gamma_\pi\gamma_{\pi'}\theta_1\cdots\theta_n}(G)) \subseteq F_{\gamma_-\gamma_{-'}\theta_1\cdots\theta_n}(G).$$

PROOF. Let θ denote the Fitting class $\gamma_\pi\gamma_{\pi'}\theta_1\cdots\theta_n$. It is clear that $C_G(F_\theta(G))$ is also $\theta\gamma_{(\pi(G))'}$-separable and that $O_{(\pi(G))'}(G) = 1$ (For the meaning of $\pi(G)$ see [2]). By Lemmas 4.5 (i), 4.4 and 4.3 $C_G(F_\theta(G))$ has θ-length of order 1. Since $F_{\theta_{-(G)}}(G) = F_\theta(G)$, the main theorem implies the conclusion.

COROLLARY 4.7. *If $C_G(F_{(\gamma_{\pi'}\gamma_\pi)_n}(G))$ is π-separable, $n \geqq 2$, then $C_G(F_{(\gamma_{\pi'}\gamma_\pi)_n}(G)) \subseteq F_{(\gamma_{\pi'}\gamma_\pi)_n}(G)$.*

In particular for $n=2$, we obtain that if $C_G(O_\pi(G))$ is π-separable and $O_{\pi'}(G) = 1$, then $C_G(O_\pi(G)) \subseteq O_\pi(G)$. Corollary 4.7 is a generalized version of Lemma 1.2.3 of Hall-Higman [3]. In the same way, with minor changes, we obtained the next three conclusions.

COROLLARY 4.8 *If $C_G(F_{(\gamma_{\pi'}v_\pi)_n}(G))$ is π-solvable, $n \geqq 2$, then $C_G(F_{(\gamma_{\pi'}v_\pi)_n}(G)) \subseteq F_{(\gamma_{\pi'}v_\pi)_n}(G)$.*

In particular for $n = 2$ we obtain that if $C_G(F(G))$ is π-solvable and $O_{\pi'}(G) = 1$, then $C_G(F(G)) \subseteq F(G)$. (Note that $O_{\pi'}(G) = 1$ implies that $F_{v_\pi}(G) = F(G)$).

COROLLARY 4.9. *If G is π-solvable and $\bar{G} =: G/O_{\pi'}(G)$, then $C_{\bar{G}}(F(\bar{G})) \subseteq F(\bar{G})$.*

COROLLARY 4.10. *If $C_G(F_{(v_{\pi'}v_\pi)n}(G))$, $n \geqq 2$, is a solvable subgroup of a group G, then*

$$C_G(F_{(v_{\pi'}v_\pi)_n}(G)) \subseteq F_{(v_{\pi'}v_\pi)_n}(G).$$

In particular for $\pi = \pi(G)$ we obtain that if $C_G(F^n(G))$ is a solvable subgroup of a group G, then $C_G(F^n(G)) \subseteq F^n(G)$, where $F^n(G)$ is defined by induction in the following way: $F^1(G) = F(G)$ and $F^n(G)/F^{n-1}(G) = F(G/F^{n-1}(G)$. Corollaries 4.8 and 4.10 are generalized versions of the well known theorem: If G is a solvable group, then $C_G(F(G)) \subseteq F(G)$.

THEOREM 4.11. *Let G be a group and let $\bar{G} = G/O_{\pi'}(G)$. If K is an S_π-subgroup of $F_{\gamma_{\pi'}v_\pi}(G)$, then $C_G(K) \subseteq F_{\gamma_{\pi'}v_\pi}(G)$ if and only if $C_{\bar{G}}(F(\bar{G})) \subseteq F(\bar{G})$.*

PROOF. Set $N = N_G(K)$ and suppose that $C_G(K) \subseteq F_{\gamma_{\pi'}v_\pi}(G)$. It is well known that: If $H \lhd G$ and K is a nilpotent S_π-subgroup of H then $G = N_G(K)H$. Hence $G = O_{\pi'}(G)N$. Thus the natural homomorphism of G onto $\bar{G}$ maps N onto $\bar{G}$. This implies that there exists a subgroup C on N whose image is $C_{\bar{G}}(F(\bar{G}))$. Since K maps onto $F_{v_\pi}(\bar{G}) = F(\bar{G})$ and $K \lhd N$, it follows that $[C, K] \subseteq K \cap O_{\pi'}(G) = 1$. Thus $C \subseteq C_G(K) \subseteq F_{\gamma_{\pi'}v_\pi}(G)$, whence $C_{\bar{G}}(F(\bar{G}) \subseteq F(\bar{G})$.

On the other hand assume that $C_{\bar{G}}(F(\bar{G})) \subseteq F(\bar{G})$. Since K maps onto $F_{v_\pi}(\bar{G}) = F(\bar{G})$, therefore $C_G(K)$ maps into $C_{\bar{G}}(F(\bar{G}))$ and consequently $C_G(K)$ maps into $F(\bar{G})$. Hence $C_G(K) \subseteq F_{\gamma_{\pi'}v_\pi}(G)$.

THEOREM 4.12. *If K is an S_π-subgroup of the π-solvable group G, then $C_G(K \cap F_{\gamma_{\pi'}v_\pi}(G)) \subseteq F_{\gamma_{\pi'}v_\pi}(G)$. In particular $Z(K) \subseteq F_{\gamma_{\pi'}v_\pi}(G)$.*

PROOF. Set $\bar{G} = G/O_{\pi'}(G)$. Since G is π-solvable, Corollary 4.9 implies that $C_{\bar{G}}(F(\bar{G})) \subseteq F(\bar{G})$. Now, $K \cap F_{\gamma_{\pi'}v_{\pi}}(G)$ is an S_{π}-subgroup of $F_{\gamma_{\pi'}v_{\pi}}(G)$. Therefore Theorem 4.11 implies that $C_G(K \cap F_{\gamma_{\pi'}v_{\pi}}(G)) \subseteq F_{\gamma_{\pi'}v_{\pi}}(G)$. Since $Z(K)$ centralizes $K \cap F_{\gamma_{\pi'}v_{\pi}}(G)$ it follows, in particular, that $Z(K) \subseteq F_{\gamma_{\pi'}v_{\pi}}(G)$.

THEOREM 4.12 is a generalized version of the well known theorem: If P is an S_p-subgroup of the p-solvable group G, then $C_G(P \cap O_{p'p}(G)) \subseteq O_{p'p}(G)$.

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